EPFL

Week 14:
Beam Buckling

- Euler Buckling
- Effective length for buckling
- Effect of eccentricity

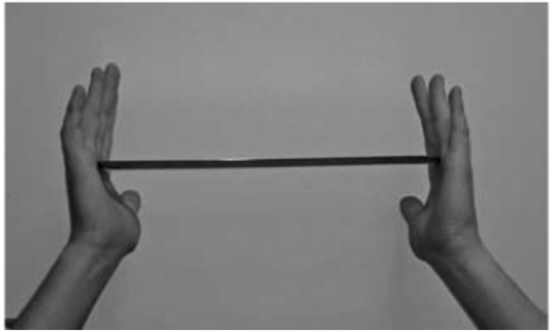
Stability- different criteria for resisting loads

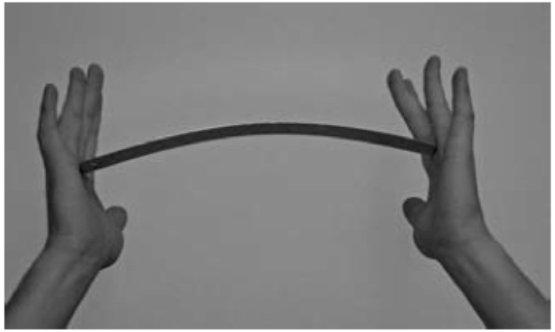
Strength: the ability of a structure to withstand a load without the development of excessive stress

Stiffness: the ability of a structure to withstand a load without developing excessive deformation.

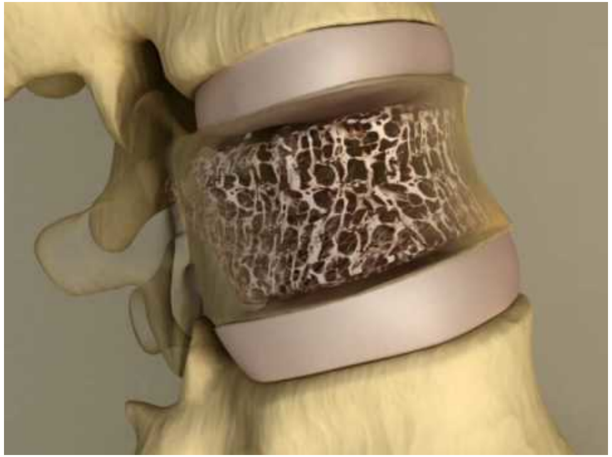
Stability: the ability of a structure to withstand a load without experiencing a sudden change in configuration

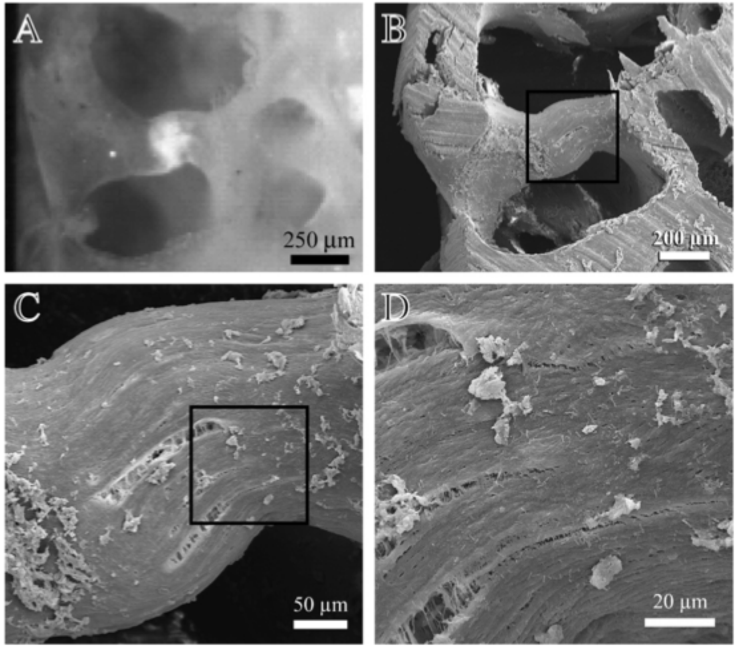
Buckling

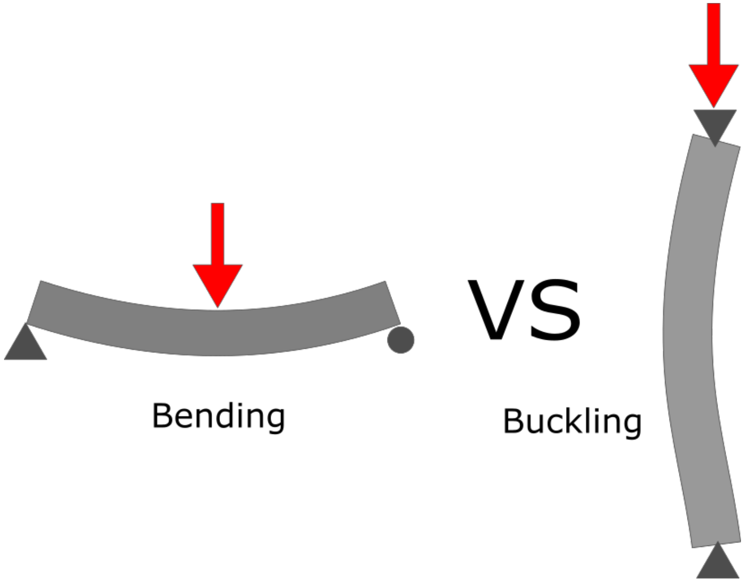




Buckling is important from the macro to the microscale

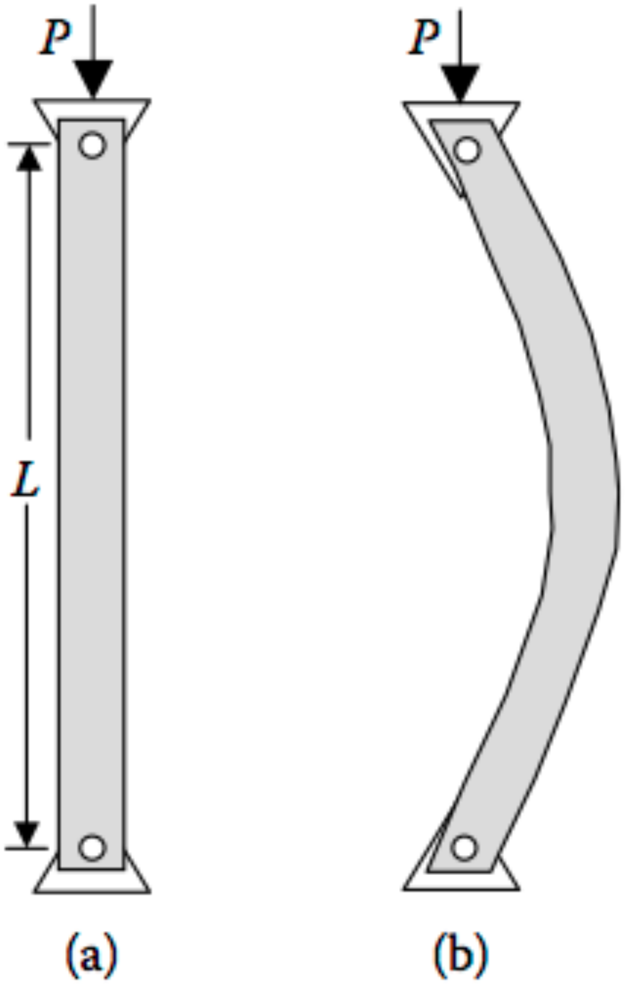






Buckling

- Buckling is a type of instability that occurs when a beam fails under a compressive load much smaller than the load necessary to reach the yield stress
- In buckling, the failure occurs because the applied load results in a sudden deformation in a perpendicular direction.

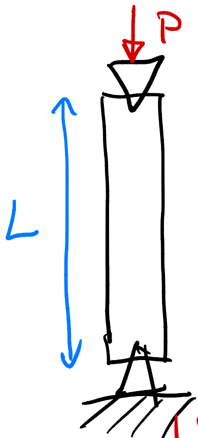


Euler Buckling

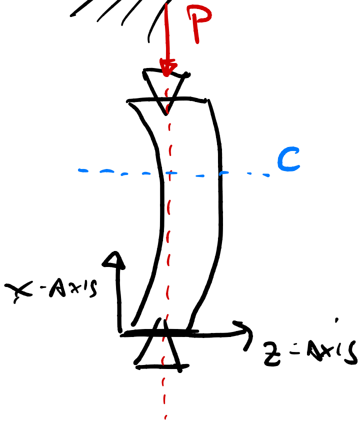
Two regimes of deformation when a beam is loaded in compression:

1. If the axial load on a beam is small, the change in length will be due to compressive strain. $P<P_{cr}$
2. If the axial load on a beam P is larger than the critical load $P_{cr,}$ then the beam becomes instable and a small perturbation will result in buckling of the beam.

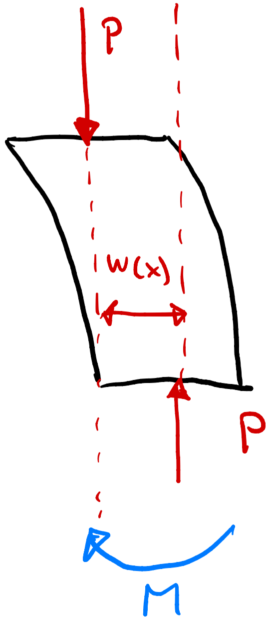
Euler Buckling:



- $P < P_{CR} \Rightarrow \Delta L = L \cdot \varepsilon = L \varepsilon \frac{P}{A}$
- $P > P_{CR} \Rightarrow$ Buckling



$\Rightarrow$ TAKE VIRTUAL SECTION THROUGH BEAM AT HEIGHT C



- DUE TO BENDING OF BEAM: THE LOAD P IS NO LONGER CO-LINEAR WITH THE REACTION LOAD ON THE VIRTUAL SECTION

  ⇒ AN INTERNAL MOMENT MUST EXIST THAT KEEPS THE SECTION FROM ROTATING

- INTERNAL MOMENT: $M(x) = P \cdot w(x)$

- WE KNOW FROM BEAM BENDING:

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} = -\frac{P}{EI} w(x)$$

- DIFF. EQN. FOR BUCKLING:

$$\boxed{\frac{d^2 w}{dx^2} + \frac{P}{EI} w = 0}$$

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- Solve with characteristic polynomial:

$$\lambda^2 + \frac{P}{EI} = 0 \Rightarrow \lambda = \pm i\sqrt{\frac{P}{EI}}$$

- The solution with: $\lambda = a \pm bi$ $\quad b = \sqrt{\frac{P}{EI}}$ $\quad a = 0$

$$w(x) = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$$

$$w(x) = c_1 \cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x$$

- c_1 & c_2 through B.C.

$$w(0) = 0 \Rightarrow c_1 = 0$$

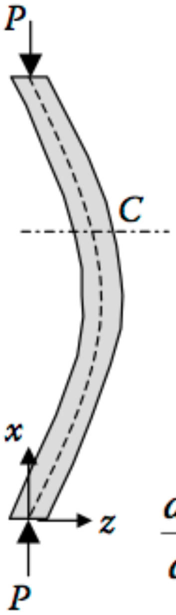
$$w(L) = 0 \Rightarrow w(L) = c_2 \sin\left(\sqrt{\frac{P}{EI}}\,L\right) = 0$$

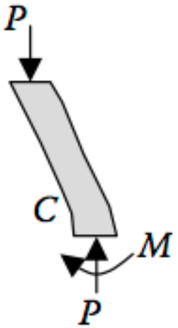
$$\sin\left(\sqrt{\frac{P}{EI}}\,L\right) = 0$$

$$\sqrt{\frac{P}{EI}}\,L = n \cdot \pi \qquad n \in \mathbb{Z}$$

$$\boxed{P = \frac{n^2 \pi^2}{L^2} EI}$$

Euler Buckling

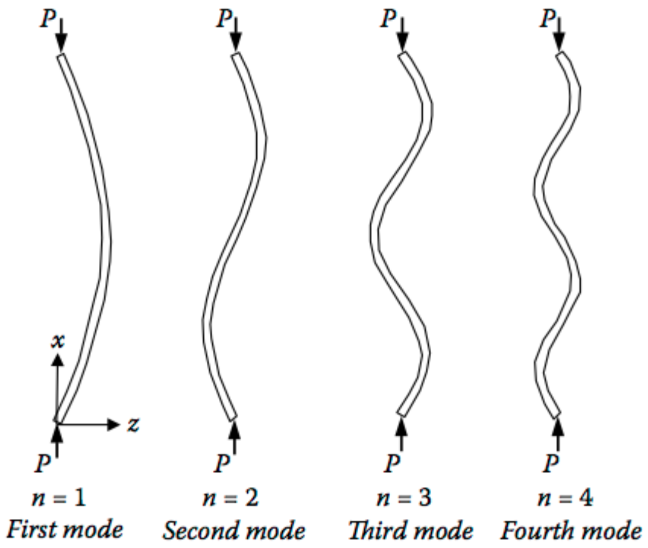




$$\frac{d^2w}{dx^2} = -\frac{M}{EI} = -\frac{P}{EI}w$$

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0$$

We can derive the Euler Buckling formula using the method of sections through the buckled beam.



Euler Buckling

The differential equation has multiple solutions:

This results in multiple bucking modes:

$$P = \frac{n^2 \pi^2 EI}{L^2}$$

Euler Formula

- For the critical buckling load we get then *Euler's Buckling Formula*:

$$P_{cr} = \frac{\pi^2 EI}{L^2} .$$

- With the shape of the buckled beam:

$$w(x) = a\sin(\frac{\pi}{L}x)$$

$$w(x) = c_2 \sin\left(\sqrt{\frac{P_{cr}}{EI}}\, x\right) = c_2 \sin\left(\frac{\pi}{L} x\right)$$

- The second moment of area (I) should be taken around the axis around which the beam buckles. This is in general the axis with the smallest second moment of area.

Euler Buckling : Effective length

$$P_{cr} = \frac{\pi^2 EI}{L_e^2}$$

Effective Length L_e as Function of Supports

End Conditions	Effective Length
Fixed-Free	$Le = 2L$
Pinned-Pinned	$Le = L$
Fixed-Pinned	$Le = 0.7L$
Fixed-Fixed	$Le = 0.5L$

- The Euler Formula we have derived here only deals with a beam with pinned supports on each end.
- The type of support however greatly influences the critical load and the buckling behavior.
- Euler's formula can be extended towards other types of support by using the concept of the effective length.

Critical Buckling stress

- Critical buckling stress:

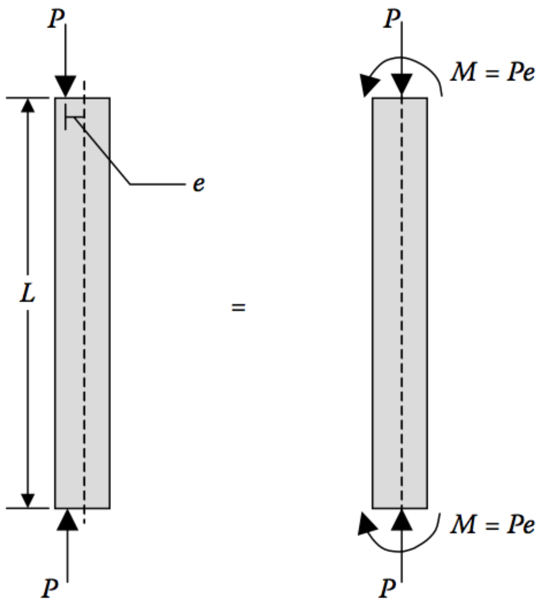
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{L_e^2 A}$$

- Using the definition of the radius of gyration $r = \sqrt{\frac{I}{A}}$:

$$\sigma_{cr} = \frac{\pi^2 EAr^2}{L_e^2 A} = \frac{\pi^2 Er^2}{L_e^2} = \frac{\pi^2 E}{(L_e / r)^2}$$

- L_e/r is the slenderness ratio

$$w(x) = c_2 \sin\left(\frac{\pi}{L} x\right)$$



Buckling: Effect of eccentricity

- So far, we've looked at beams that were loaded through the centroid of the column
- Often, the load is offset from this axis: eccentric loading
- We calculate the behavior of a beam pinned at both ends with eccentric load.
- We can model the eccentricity with an axial load and a moment at the supports

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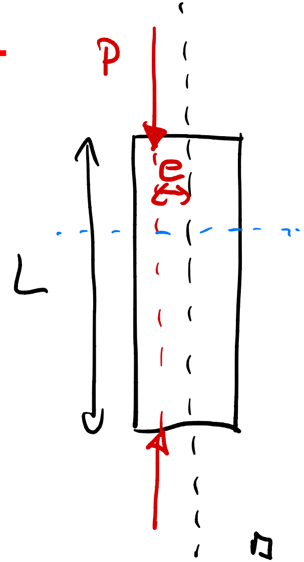

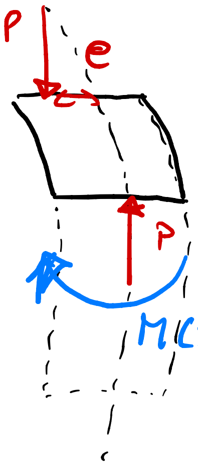

$$M(x) = P \cdot w + Pe$$

$$\frac{d^2 w}{dx^2} = -\frac{M}{EI} = -\frac{P}{EI} w - \frac{P}{EI} e$$

- INHOMOGENEOUS DIFFERENTIAL EQN:

$$\boxed{\frac{d^2 w}{dx^2} + \frac{P}{EI} w = -\frac{P}{EI} e}$$

- SOLVE THROUGH ANSATZ: $\underline{w(x) = a}$

$$\frac{Pa}{EI} = -\frac{P}{EI} e \Rightarrow a = -e$$

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- GENERAL SOLUTION OF THIS INHOMOGENEOUS EQN IS:

$$w(x) = c_1 \cos\left(\sqrt{\frac{P}{EI}}\, x\right) + c_2 \sin\left(\sqrt{\frac{P}{EI}}\, x\right) - e$$

- BC:

$$w(0) = 0 \quad \Rightarrow \quad c_1 = e$$

$$w(L) = 0 \quad e \cos\left(\sqrt{\frac{P}{EI}} \cdot L\right) + c_2 \sin\left(\sqrt{\frac{P}{EI}} \cdot L\right) - e = 0$$

WITH TRIGONOM. FORMULAS: $\frac{1 - \cos x}{\sin x} = \tan \frac{x}{2}$

$$c_2 = e \tan\left(\sqrt{\frac{P}{EI}} \cdot \frac{L}{2}\right)$$

⇒ FORMULA FOR DEFLECTION IN BUCKLING:

$$w(x) = e\cos\left(\sqrt{\frac{P}{EI}}\,x\right) + e\tan\left(\sqrt{\frac{P}{EI}}\,\frac{L}{2}\right)\cdot\sin\left(\sqrt{\frac{P}{EI}}\,x\right) - e$$

□ MAXIMUM DEFLECTION IS AT $x = \frac{L}{2}$ FOR $P < P_{CR}$

$$w_{MAX} = e\left\{\sec\left(\sqrt{\frac{P}{EI}}\,\frac{L}{2}\right) - 1\right\}$$

$$\sec = \frac{1}{\cos}$$

□ BUCKLING OCCURS AT P_{CR} WITH $w = \infty$

$$\sec\left(\frac{\pi}{2}\right) = \infty \Rightarrow \sqrt{\frac{P_{CR}}{EI}}\cdot\frac{L}{2} = \frac{\pi}{2}$$

$$\Rightarrow P_{CR} = \frac{\pi^2 EI}{L^2}$$

CRITICAL LOAD

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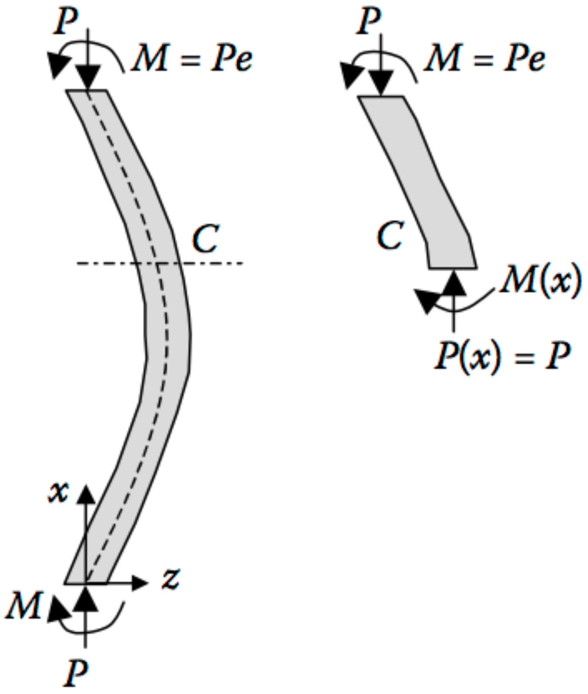
- MAXIMUM STRESS IN COLUMN BEFORE BUCKLING: $\sigma_{comp} + \sigma_{BENDING}$

$$\sigma_{MAX} = -\frac{P}{A}\left\{1 - \frac{e \cdot c}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right\}$$

$$r = \sqrt{\frac{I}{A}}$$

c = MAXIMUM DISTANCE FROM CENTROID TO SURFACE



Buckling: Effect of eccentricity

- Centric load: P
- Moment: M=P*e
- This means that the beam bends even under small loads without the beam buckling
- We solve the now inhomogeneous differential equation:

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = -\frac{P}{EI} \cdot e$$

Buckling: Effect of eccentricity

- For the deflection we then get:

$$w(x) = e\left\{\tan\left(\sqrt{\frac{P}{EI}}\cdot\frac{L}{2}\right)\cdot\sin\left(\sqrt{\frac{P}{EI}}\cdot x\right)+\cos\left(\sqrt{\frac{P}{EI}}\cdot x\right)-1\right\}$$

- And for the maximum deflection:

$$w_{max} = w(L/2) = e\left[\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)-1\right]$$

- The critical bucking load is then

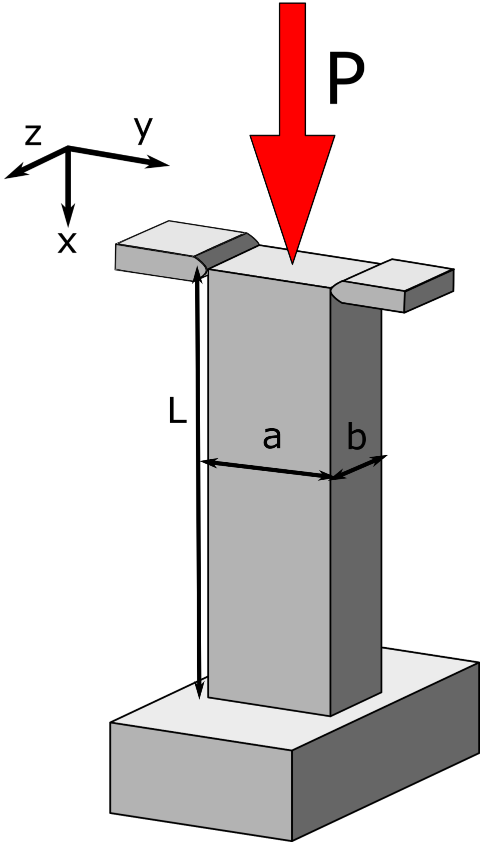
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Buckling: Effect of eccentricity

- Maximum stress in the beam is given by the compressive stress and the bending stress:

$$\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\right]$$

$$\sigma_{max} = -\frac{P}{A}\left[1 - \frac{ec}{r^2}\sec\left(\sqrt{\frac{P}{P_{cr}}}\frac{\pi}{2}\right)\right]$$



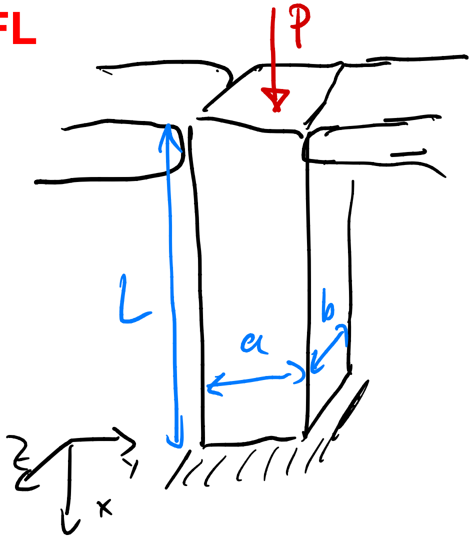
Example Buckling

An aluminum column of length *L* and rectangular cross section has a fixed end *B* and supports a centric axial load at *A*.

Two smooth and rounded fixed plates restrain end *A* from moving in one of the vertical planes of symmetry but allow it to move in the other plane.

(a) Determine the ratio *a/b* of the two sides of the cross section corresponding to the most efficient design against buckling.

(b) Design the most efficient cross section for the column, knowing that *L* = 50 cm, *E* = 70 GPa, *P* = 22 kN and that a safety factor of 2.5 is required.



- **Given:**
  - Geometry: $L = 50\ cm$
  - Loads: $P = 22\ kN$
  - Mat prop: $E = 70\ GPa$
  - Safety Fact: 2.5
- **Asked:**
  - a/b for largest total P_{crit} @ $A = a \cdot b$
  - A_{min} to withstand $P = 22\ kN$ @ $SF = 2.5$
- **Approach:** Buckling with different B.C.

- **Answer:**
  - Buckling possible in xy & xz plane
  - xy & xz have different supports

    xy: fixed – pinned support

    xz: fixed – free

- MOST EFFICIENT DESIGN: $P_{cr}^{xy} = P_{cr}^{xz}$

$$P_{cr} = \frac{\pi^2 E I}{L_e^2}$$

$$P_{cr}^{xy} = \frac{\pi^2 E I_z}{(L_e^{xy})^2} = \frac{\pi^2 E I_y}{(L_e^{xz})^2} = P_{cr}^{xz}$$

$$I_z = \frac{b a^3}{12} \qquad I_y = \frac{a b^3}{12}$$

$L_e^{xy} = 0{,}7 L$ ... FIXED-PINNED $\qquad L_e^{xz} = 2L$ ... FIXED-FREE

$$\frac{b a^3}{(0{,}7)^2} = \frac{a b^3}{2^2} \Rightarrow \frac{a^2}{b^2} = \frac{0{,}7^2}{4} \Rightarrow \boxed{\frac{a}{b} = 0{,}35} \Rightarrow a = 0{,}35 \cdot b$$

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ANSWER TO PART b:

$P_{cR} = 2.5 \cdot P = 55\,kN$

$$P_{cR} = \frac{\pi^2 E I}{L_e^2}$$

$I_y = \frac{ab^3}{12}$ $\quad a = 0.35\,b$

$L_e = 2L$ $\quad E = 70\,GPa$

$$= \frac{\pi^2 E}{L_e^2} \cdot \frac{ab^3}{12} = 55\,kN$$

$$= \frac{\pi^2 E}{48L^2} \cdot 0{,}35\,b^4 = 55\,kN$$

$$\Rightarrow b^4 = \frac{48L^2}{0{,}35 \cdot \pi^2 E} \cdot 55\,kN$$

$\Rightarrow b = 4{,}06\,cm \Rightarrow$ <u>4,1 cm</u>

<u>$a = 1.4\,cm$</u>